

# Solution of Matrix Eigenvalue Problem

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# Standard Matrix Eigenvalue Problem: Introduction

Consider a system of equations in algebraic form

$$\begin{aligned}(a_{11} - \lambda)x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + (a_{22} - \lambda)x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + (a_{nn} - \lambda)x_n &= 0\end{aligned}$$

This is not a normal system of linear algebraic equations we're used to. For one, there are  $n$  equations, but  $n + 1$  unknowns (the  $x_i$  values, and also  $\lambda$ ). This particular system of equations is known as *the standard eigenvalue problem*.

## Form 1

The three forms shown are all algebraically equivalent. Any system of equations that can be expressed in these forms is a standard eigenvalue problem.

$$\begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{Bmatrix}$$

## Form 2

$$\left( \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \right) \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{Bmatrix}$$

$$([A] - \lambda [I]) \{x\} = \{0\}$$

# Form 3

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = \lambda \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}$$

$$[A] \{x\} = \lambda \{x\}$$

## Solvability of the Standard Eigenvalue Problem

Recall form 2 of the standard eigenvalue problem:

$$([A] - \lambda [I]) \{x\} = \{0\}$$

This system of equations has a solution for values of  $\lambda$  that cause the determinant of the coefficient matrix to equal 0, that is:

$$|[A] - \lambda [I]| = 0$$

# Characteristic Equation

Expanding out all the terms of the previous determinant

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0$$

yields a long polynomial in terms of  $\lambda$ . This polynomial will be  $n$ th order, and will therefore have  $n$  roots, each of which may be real or complex.

# General Eigenvalue Problem: Introduction

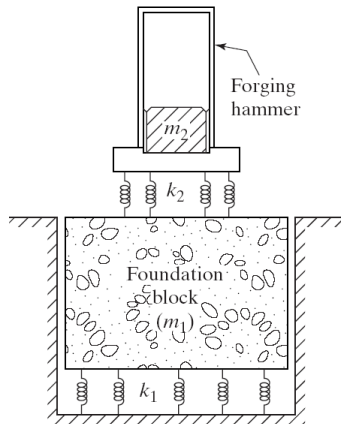
Many physical systems do not automatically present themselves as a standard eigenvalue problem, even though they can be reformatted as a standard eigenvalue problem. The form of a *general eigenvalue problem* is

$$[A] \{x\} = \lambda [B] \{x\}$$

where  $[A]$  and  $[B]$  are symmetric matrices of size  $n \times n$ .

# General Eigenvalue Problem Example

A forging hammer of mass  $m_2$  is mounted on a concrete foundation block of mass  $m_1$ . The stiffnesses of the springs underneath the forging hammer and the foundation block are given by  $k_2$  and  $k_1$ , respectively.



# General Eigenvalue Problem Example

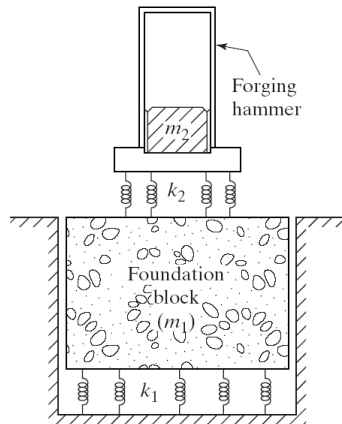
The system undergoes simple harmonic motion at one of its natural frequencies  $\omega$ . That is:

$$x_1(t) = \cos(\omega t + \phi_1)$$

$$x_2(t) = \cos(\omega t + \phi_2)$$

$$a_1(t) = -\omega^2 x_1(t)$$

$$a_2(t) = -\omega^2 x_2(t)$$



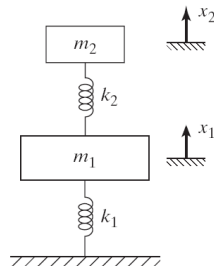
# General Eigenvalue Problem Example

Each mass in the system obeys Newton's second law of motion, that is:

$$\Sigma F = ma$$

Forces on the foundation block:

- forces from the lower springs, which counteracts motion in the  $x$  direction at an amount  $-k_1x_1$
- forces from the upper springs, which act according to the amount of relative displacement of the masses  $m_1$  and  $m_2$ :  
 $-k_2(x_1 - x_2)$



# General Eigenvalue Problem Example

The equilibrium equation for the foundation mass is then

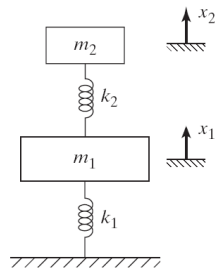
$$\Sigma F = ma$$

$$-k_1x_1 - k_2(x_1 - x_2) = m_1 a$$

$$(-k_2 - k_1)x_1 + k_2x_2 = m_1 a$$

$$(-k_2 - k_1)x_1 + k_2x_2 = -m_1\omega^2 x_1$$

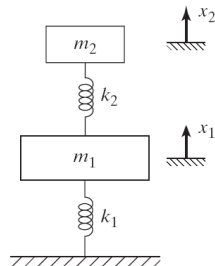
$$(k_1 + k_2)x_1 - k_2x_2 = m_1\omega^2 x_1$$



# General Eigenvalue Problem Example

Similarly, the equilibrium equation for the forging hammer mass is

$$-k_2 x_1 + k_2 x_2 = m_2 \omega^2 x_2$$



# General Eigenvalue Problem Example

So the two equations of motion are

$$\begin{aligned}(k_1 + k_2)x_1 - k_2x_2 &= m_1\omega^2x_1 \\ -k_2x_1 + k_2x_2 &= m_2\omega^2x_2\end{aligned}$$

or in matrix form

$$\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \omega^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

This is a general eigenvalue problem

$$[A]\{x\} = \lambda[B]\{x\}$$

where  $[A]$  is the spring matrix,  $\{x\}$  is the vector of  $x$  values,  $\lambda$  is  $\omega^2$ , and  $[B]$  is the mass matrix.

# Eigenvalue Solutions in MATLAB: Standard Problems

The design of a mechanical component requires that the maximum principal stress to be less than the material strength. For a component subjected to arbitrary loads, the principal stresses  $\sigma$  are given by the solution of the equation

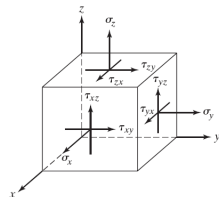
$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} \begin{Bmatrix} l_x \\ l_y \\ l_z \end{Bmatrix} = \sigma \begin{Bmatrix} l_x \\ l_y \\ l_z \end{Bmatrix}$$

where the  $\sigma$  values represent normal stresses in the  $x$ ,  $y$ , and  $z$  directions, and the  $\tau$  values represent shear stresses in the  $xy$ ,  $xz$ , and  $yz$  planes. The  $l$  values represent direction cosines that define the principal planes on which the principal stress occurs.

# Eigenvalue Solutions in MATLAB: Standard Problems

Determine the principal stresses and principal planes in a machine component for the following stress condition

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} 10 & 4 & -6 \\ 4 & -6 & 8 \\ -6 & 8 & 14 \end{bmatrix} \text{ MPa}$$



## MATLAB Solution

```
clear all
sigma=[10  4 -6
        4 -6  8
        -6  8 14];
[dirs, stresses]=eig(sigma);
% diag(A) extracts the elements of the
% [A] matrix along the diagonal
principalStressList=diag(stresses)
principalDirs=dirs
```

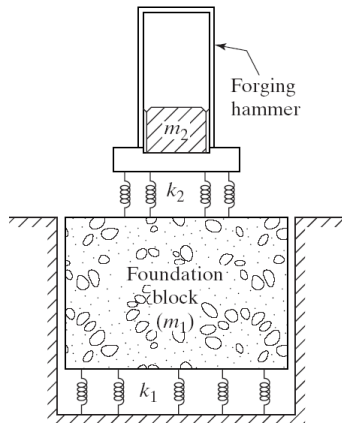
## MATLAB Solution

```
>> rao_p431
principalStressList =
    -10.4828
     9.3181
    19.1647
principalDirs =
    -0.2792    0.8343   -0.4754
     0.8905    0.4102    0.1970
    -0.3594    0.3683    0.8574
```

# Eigenvalue Solutions in MATLAB: General Problems

Solve the forging hammer problem for the following values:

- $m_1 = 20000$  kg
- $m_2 = 5000$  kg
- $k_1 = 1 \times 10^7$  N/m
- $k_2 = 5 \times 10^6$  N/m



# Solving Eigenvalue Problems in MATLAB

Solving this eigenvalue problem will yield 2 eigenvalues equal to the square of the system's natural frequencies, and 2 corresponding  $x$  vector values that show the relative displacements of the  $m_1$  and  $m_2$  masses at those frequencies.

## MATLAB Solution (Part 1)

```
clear all;  
% Define spring constants and masses  
% for hammer and foundation block  
k1=1e7;  
k2=5e6;  
m1=20000;  
m2=5000;  
  
% Define system stiffness matrix  
K=[k1+k2 -k2  
    -k2  k2];  
% Define system mass matrix  
M=[m1  0  
    0  m2];
```

## MATLAB Solution (Part 2)

```
% Solve general eigenvalue problem  
[X,Omega2]=eig(K,M);  
% diag(A) extracts the elements of the  
% [A] matrix along the diagonal  
Omega=diag(sqrt(Omega2));  
% Scale column 1 of the [X] matrix by  
% the row 1, column 1 X value  
X(:,1)=X(:,1)/X(1,1);  
% Scale column 2 of the [X] matrix by  
% the row 1, column 2 X value  
X(:,2)=X(:,2)/X(1,2);
```

```
Omega  
X
```

## MATLAB Solution (Results)

```
>> rao_ex42  
Omega =  
    18.9634  
    37.2879  
X =  
    1.0000    1.0000  
    1.5616   -2.5616
```