A Novel Single-step Approach for Self-coherent Tomography Using Semidefinite Relaxation

Zhen Hu*, Robert C. Qiu*, James P. Browning†, Michael Wicks‡

* Cognitive Radio Institute, Department of Electrical and Computer Engineering, Center for Manufacturing Research, Tennessee Technological University, Cookeville, Tennessee 38501, USA
† Air Force Research Laboratory, Wright Paterson AFB, OH 45433, USA
‡ Sensor Systems Division, University of Dayton Research Institute Dayton, Ohio 45469, USA

Abstract—This letter will present a novel single-step approach for self-coherent tomography using semidefinite relaxation. Phase retrieval for scattered fields is not required. The general solver can be used to solve the corresponding convex optimization problem and image the target. Both man-made and experimental data will be exploited to demonstrate the performance of the proposed approach. The imaging results illustrate the benefit of bringing the state of the art mathematics to inverse scattering or diffraction tomography.

Index Terms—Wireless Tomography, Self-coherent Tomography, Inverse Scattering, Phase Retrieval, Convex Optimization, Semidefinite Programming, Relaxation.

I. INTRODUCTION

As smart phones are widely used, there will be a potential large-scale wireless communication network deployed around the world. Remote sensing can be embedded into this kind of large-scale wireless communication network. However, wireless communication devices are not specifically designed for remote sensing. These devices do not meet the high-accuracy measurement requirements of remote sensing. For example, inverse scattering or diffraction tomography requires accurate phase information to perform imaging. Accurate phase information is hard to obtain using commercial wireless communication devices, especially in the presence of noise. Thus, self-coherent tomography in the framework of wireless tomography was proposed [1]. Wireless tomography which combines wireless communication and radio frequency (RF) tomography gives a novel approach to remote sensing.

In self-coherent tomography, we do not need to measure phase information of total or scattered fields. However, the full-data scattered fields should be reconstructed given the full-data incident fields and the amplitude-only total fields [2]–[5]. This step is called phase retrieval for scattered fields. Then, coherent tomography is performed.

Meanwhile, the single-step approach can also be applied to amplitude-only inverse scattering once the full-data incident fields are known [4]. A subspace-based optimization method for inverse scattering utilizing the amplitude-only total fields has been developed in [6]. The scatterer’s permittivity profile is reconstructed by using only intensity data of the total fields with no phase information [6]. The distorted Rytov iterative method with phaseless data is also used for tomographic reconstruction [7]. Phase retrieval for scattered fields is not required.

This letter will explore a novel single-step approach for self-coherent tomography using semidefinite relaxation [8], [9]. No phase retrieval for scattered fields is required. The general solver to the convex optimization problem can be used. The similar idea can be found in [10]. However, mutual multi-scattering is negligible [10]. There is no incident field considered in array imaging using intensity-only measurements [10]. Hence, the corresponding problem is homogeneous quadratically constrained quadratic programming (QCQP). There is a phase ambiguity problem. Only man-made data with sparse scatterers of real-valued reflectivities are used to verify the proposed approach [10]. In this letter, we will consider both mutual multi-scattering and incident field in inverse scattering or diffraction tomography. The general solution to self-coherent tomography will be given. Both man-made and experimental data will be exploited to demonstrate the performance of the novel single-step approach.

The rest of the letter is organized as follows. In section II, the system model of self-coherent tomography is given. Mathematical background of convex optimization and semidefinite relaxation is introduced in section III. The solution to self-coherent tomography is presented in section IV. Simulation results are presented in section V. Some remarks are given in section VI.

II. SELF-COHERENT TOMOGRAPHY

In self-coherent tomography, we only know amplitude-only total fields and the full-data incident fields. Accurate data will be considered in this paper without consideration of noise.

The system model in 2-D near field configuration of self-coherent tomography, which is similar to geometry of the problem shown in Fig. 1 of [3], can be described as follows. There are \( N_t \) transmitter sensors on the source domain with locations \( \Gamma_{nt}, n_t = 1,2,...,N_t \). There are \( N_r \) receiver sensors on the measurement domain with locations \( \Gamma_{nr}, n_r = 1,2,...,N_r \).
1, 2, ..., N_T. The target domain Ω is discretized into a total number of N_d subareas with the center of the subarea located at l_{n_d}^d, n_d = 1, 2, ..., N_d. The corresponding target scattering strength is \( \tau_{n_d}, n_d = 1, 2, ..., N_d \). If the \( n_1^{th} \) sensor sounds the target domain and the \( n_1^{th} \) sensor receives the amplitude-only total field, the full-data measurement equation is shown as,

\[
E_{\text{tot}}(l_{n_1}^d \rightarrow l_{n_1}^r) = E_{\text{inc}}(l_{n_1}^d \rightarrow l_{n_1}^r) + E_{\text{scatter}}(l_{n_1}^d \rightarrow \Omega \rightarrow l_{n_1}^r)
\]  

(1)

where \( |E_{\text{tot}}(l_{n_1}^d \rightarrow l_{n_1}^r)| \) is the amplitude-only total field of the \( n_1^{th} \) receiver sensor due to the sounding signal from the \( n_1^{th} \) transmitter sensor; \( E_{\text{inc}}(l_{n_1}^d \rightarrow l_{n_1}^r) \) is the incident field directly from the \( n_1^{th} \) transmitter sensor to the \( n_1^{th} \) receiver sensor; \( E_{\text{scatter}}(l_{n_1}^d \rightarrow l_{n_1}^r) \) is the scattered field from target domain which can be expressed in Eqn. (2).

In Eqn. (2), \( G(l_{n_1}^d \rightarrow l_{n_1}^r) \) is the wave propagation Green’s function from location \( l_{n_1}^d \) to location \( l_{n_1}^r \) and \( E_{\text{tot}}(l_{n_1}^d \rightarrow l_{n_1}^r) \) is the total field in the target subarea \( l_{n_1}^d \) caused by the sounding signal from the \( n_1^{th} \) transmitter sensor which can be represented as the state equation shown in Eqn. (3).

Hence, the goal of self-coherent tomography is to recover \( \tau_{n_d}, n_d = 1, 2, ..., N_d \) and image the target domain \( \Omega \) based on Eqn. (1), Eqn. (2), and Eqn. (3).

Define \( e_{\text{tot},m} \in R^{N_{N_t} N_r \times 1} \) as

\[
e_{\text{tot},m} = \begin{bmatrix}
|E_{\text{tot}}(l_{n_1}^d \rightarrow l_{n_1}^r)| \\
|E_{\text{tot}}(l_{n_2}^d \rightarrow l_{n_2}^r)| \\
\vdots \\
|E_{\text{tot}}(l_{N_T}^d \rightarrow l_{N_T}^r)| \\
\end{bmatrix}.
\]  

(4)

Define \( e_{\text{inc},m} \in C^{N_{N_t} N_r \times 1} \) as

\[
e_{\text{inc},m} = \begin{bmatrix}
E_{\text{inc}}(l_{n_1}^d \rightarrow l_{n_1}^r) \\
E_{\text{inc}}(l_{n_2}^d \rightarrow l_{n_2}^r) \\
\vdots \\
E_{\text{inc}}(l_{N_T}^d \rightarrow l_{N_T}^r) \\
\end{bmatrix}.
\]  

(5)

Define \( e_{\text{scatter},m} \in C^{N_{N_t} N_r \times 1} \) as

\[
e_{\text{scatter},m} = \begin{bmatrix}
e_{\text{scatter},m,1} \\
e_{\text{scatter},m,2} \\
\vdots \\
e_{\text{scatter},m,N_T} \\
\end{bmatrix}
\]  

(6)

where based on Eqn. (2) \( e_{\text{scatter},m,n_1} \in C^{N_r \times 1} \) is described as,

\[
e_{\text{scatter},m,n_1} = G_m E_{\text{tot},s,n_1} \tau
\]  

(7)

where \( G_m \in C^{N_r \times N_d} \) is defined as

\[
G_m = \begin{bmatrix}
G(l_{n_1}^d \rightarrow l_{n_1}^r) & G(l_{n_2}^d \rightarrow l_{n_2}^r) & \cdots & G(l_{N_T}^d \rightarrow l_{N_T}^r) \\
G(l_{n_1}^d \rightarrow l_{n_2}^r) & G(l_{n_2}^d \rightarrow l_{n_2}^r) & \cdots & G(l_{N_T}^d \rightarrow l_{N_T}^r) \\
\vdots & \vdots & \ddots & \vdots \\
G(l_{n_1}^d \rightarrow l_{N_T}^r) & G(l_{n_2}^d \rightarrow l_{N_T}^r) & \cdots & G(l_{N_T}^d \rightarrow l_{N_T}^r) \\
\end{bmatrix}
\]  

(8)

and \( \tau \) is

\[
\tau = \begin{bmatrix}
\tau_1 \\
\tau_2 \\
\vdots \\
\tau_{N_d} \\
\end{bmatrix}.
\]  

(9)

Besides, \( E_{\text{tot},s,n_1} = \text{diag}(e_{\text{tot},s,n_1}) \) and \( e_{\text{tot},s,n_1} \in C^{N_d \times 1} \) can be expressed as based on Eqn. (3),

\[
e_{\text{tot},s,n_1} = (I - G_s \text{diag}(\tau))^{-1} e_{\text{inc},s,n_1}
\]  

(10)

where \( G_s \in C^{N_d \times N_d} \) is defined as

\[
G_s = \begin{bmatrix}
0 & G(l_{n_1}^d \rightarrow l_{n_1}^r) & \cdots & G(l_{n_1}^d \rightarrow l_{n_T}^r) \\
G(l_{n_2}^d \rightarrow l_{n_2}^r) & 0 & \cdots & G(l_{n_2}^d \rightarrow l_{n_T}^r) \\
\vdots & \vdots & \ddots & \vdots \\
G(l_{N_T}^d \rightarrow l_{n_1}^r) & G(l_{N_T}^d \rightarrow l_{n_2}^r) & \cdots & 0 \\
\end{bmatrix}
\]  

(11)

and \( e_{\text{inc},s,n_1} \in C^{N_d \times 1} \) is

\[
e_{\text{inc},s,n_1} = \begin{bmatrix}
E_{\text{inc}}(l_{n_1}^d \rightarrow l_{n_1}^r) \\
E_{\text{inc}}(l_{n_2}^d \rightarrow l_{n_2}^r) \\
\vdots \\
E_{\text{inc}}(l_{N_T}^d \rightarrow l_{N_T}^r) \\
\end{bmatrix}.
\]  

(12)

Define \( E_{\text{tot},s} \in C^{N_d N_T \times N_d} \) as

\[
E_{\text{tot},s} = \begin{bmatrix}
\text{diag}(I - G_s \text{diag}(\tau))^{-1} e_{\text{inc},s,1} \\
\text{diag}(I - G_s \text{diag}(\tau))^{-1} e_{\text{inc},s,2} \\
\vdots \\
\text{diag}(I - G_s \text{diag}(\tau))^{-1} e_{\text{inc},s,N_T} \\
\end{bmatrix}.
\]  

(13)

Define \( B_m \in C^{N_d N_T \times N_d} \) as

\[
B_m = \begin{bmatrix}
G_m & 0 & \cdots & 0 \\
0 & G_m & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & G_m \\
\end{bmatrix} E_{\text{tot},s}.
\]  

(14)

From Eqn. (4) to Eqn. (14), we can safely express \( e_{\text{tot},m} \) as

\[
e_{\text{tot},m} = [e_{\text{inc},m} + B_m \tau].
\]  

(15)

III. MATHEMATICAL BACKGROUND

Optimization stems from human instinct. We always would like to do something best. Relying on mathematics, this human instinct can be written down by mathematical optimization. Convex optimization is a subfield of mathematical optimization, which solves the problem of minimizing convex objective function based on a compact convex set. The strength of convex optimization is if a local minimum exists, then it
\[ E_{\text{scatter}} \left( \mathbf{1}_{n_t} \rightarrow \mathbf{1}_r \right) = \sum_{n_d = 1}^{N_d} G \left( \mathbf{1}_{n_d} \rightarrow \mathbf{1}_{n_t} \right) E_{\text{tot}} \left( \mathbf{1}_{n_t} \rightarrow \mathbf{1}_{n_d} \right) \tau_{n_d} \] (2)

\[ E_{\text{tot}} \left( \mathbf{1}_{n_t} \rightarrow \mathbf{1}_{n_d}^d \right) = E_{\text{inc}} \left( \mathbf{1}_{n_t} \rightarrow \mathbf{1}_{n_d}^d \right) + \sum_{n_{d'} = 1, n_{d'} \neq n_d}^{N_d} G \left( \mathbf{1}_{n_{d'}} \rightarrow \mathbf{1}_{n_d} \right) E_{\text{tot}} \left( \mathbf{1}_{n_t} \rightarrow \mathbf{1}_{n_{d'}} \right) \tau_{n_{d'}} \] (3)

is a global minimum. Hence, if the practical problem can be formulated as convex optimization problem, then global optimum can be obtained. That is one reason why convex optimization has recently become popular. Convex optimization includes the well-known linear programming, second order cone programming (SOCP), semidefinite programming (SDP), geometric programming, and so on [11].

The generalized phase retrieval problem and the corresponding approach using convex optimization have been discussed in [12]–[14]. In a linear model \( y = Ax \) where \( y \in \mathbb{C}^{M \times 1} \), \( A \in \mathbb{C}^{M \times m} \), and \( x \in \mathbb{C}^{m \times 1} \), only the squared magnitude of the output \( y \) is observed,

\[ o_i = |y_i|^2 = |a_i x|^2, \quad i = 1, 2, \ldots, M \] (16)

where

\[ A = [a_1^H \ a_2^H \ \ldots \ a_M^H]^H \] (17)

\[ y = [y_1^H \ y_2^H \ \ldots \ y_M^H]^H \] (18)

and

\[ o = [o_1^T \ o_2^T \ \ldots \ o_M^T]^T. \] (19)

where \( H \) is Hermitian operator and \( T \) is transpose operator.

We assume \( \{o_i, a_i\}_{i=1}^M \) are known and seek \( x \) which is called the generalized phase retrieval problem. Derivation from Eqn. (16) to get,

\[ o_i = a_i x (a_i x)^H \]

\[ = a_i x x^H a_i^H \]

\[ = \text{trace}(a_i^H a_i x x^H) \] (20)

where \text{trace} returns the trace value of matrix. Define \( A_i = a_i^H a_i \) and \( X = xx^H \). Both \( A_i \) and \( X \) are rank-1 positive semidefinite matrices. Then,

\[ o_i = \text{trace}(A_i X) \] (21)

which is called semidefinite relaxation.

In order to seek \( x \), we can first obtain the rank-1 positive semidefinite matrix \( X \) which can be the solution to the following optimization problem

\[
\begin{align*}
\text{minimize} & \quad \text{rank}(X) \\
\text{subject to} & \quad o_i = \text{trace}(A_i X), \quad i = 1, 2, \ldots, M \\
& \quad X \succeq 0
\end{align*}
\] (22)

However, the rank function is not a convex function and the optimization problem (22) is not a convex optimization problem. Hence, the rank function is relaxed to the trace function or the nuclear norm function which is a convex function. The optimization problem (22) can be relaxed to an SDP,

\[
\begin{align*}
\text{minimize} & \quad \text{trace}(X) \\
\text{subject to} & \quad o_i = \text{trace}(A_i X), \quad i = 1, 2, \ldots, M \\
& \quad X \succeq 0
\end{align*}
\] (23)

which can be solved by CVX which is a Matlab-based modeling system for convex optimization [15]. If the solution \( X \) to the optimization problem (23) is a rank-1 matrix, then the optimal solution \( x \) to the original phase retrieval problem is achieved by eigen-decomposition of \( X \). However, there is still a phase ambiguity problem. When the number of measurements \( M \) is fewer than necessary for a unique solution, additional assumptions are needed to select one of the solutions [14]. Motivated by compressive sensing, if we would like to seek the sparse vector \( x \), the objective function in SDP (23) can be replaced by \( \text{trace}(X) + \delta \|X\|_1 \) where \( \| \cdot \|_1 \) returns the \( l_1 \) norm of matrix and \( \delta \) is a design parameter [14].

IV. THE SOLUTION TO SELF-COHERENT TOMOGRAPHY

In this section, the solution to the linearized self-coherent tomography will be given first. Then, a novel single-step approach based on Born iterative method will be proposed to deal with self-coherent tomography with consideration of mutual multi-scattering. Distorted wave born approximation (DWBA) is used here to linearize self-coherent tomography. Specifically speaking, all the scattering within the target domain will be ignored in DWBA [16], [17]. Hence, \( E_{\text{tot}} \left( \mathbf{1}_{n_t} \rightarrow \mathbf{1}_{n_d}^d \right) \) in Eqn. (3) is reduced to \( E_{\text{tot}} \left( \mathbf{1}_{n_t} \rightarrow \mathbf{1}_{n_d}^d \right) = E_{\text{inc}} \left( \mathbf{1}_{n_t} \rightarrow \mathbf{1}_{n_d}^d \right) \) and \( B_m \) in Eqn. (14) is simplified as,

\[
B_m = \begin{bmatrix}
G_m & 0 & \cdots & 0 \\
0 & G_m & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & G_m
\end{bmatrix} \begin{bmatrix}
\text{diag}(e_{\text{inc}, s, 1}) \\
\text{diag}(e_{\text{inc}, s, 2}) \\
\vdots \\
\text{diag}(e_{\text{inc}, s, N_i})
\end{bmatrix}.
\] (24)

In this way, \( B_m \) is independent of \( \tau \) and can be calculated through Green’s function. The goal of the linearized self-coherent tomography is to recover \( \tau \) given \( e_{\text{tot}, m}, e_{\text{inc}, m} \), and \( B_m \) based on Eqn. (15).

Let \( o = e_{\text{tot}, m}; \ c = e_{\text{inc}, m}; \ A = B_m; \) and \( x = \tau \). Eqn. (15) is equivalent to

\[ o = |c + A x| \] (25)
and
\[ o_i = \text{trace}(A_i X) + |c_i|^2 + (a_i x)^* c_i + (a_i x)^* c_i \] (26)
where \( * \) returns the conjugate value of the complex number. There are two unknown variables \( X \) and \( x \) in Eqn. (26) which is different from Eqn. (21) where there is only one unknown variable \( X \). In order to solve a set of non-linear equations in Eqn. (25) to get \( x \), the following SDP is proposed,
\[
\begin{align*}
\text{minimize} \\
\text{subject to} \\
\end{align*}
\[ o_i = \text{trace}(A_i X) + |c_i|^2 + (a_i x)^* c_i + (a_i x)^* c_i \]
where \( \delta \) is a design parameter. The optimization solution \( x \) can be achieved without phase ambiguity. Furthermore, if we know additional prior information about \( x \), for example, the bound of the real or imaginary part of each entry in \( x \), this prior information can be put into the optimization problem (27) as linear constraints,
\[
\begin{align*}
\text{minimize} \\
\text{subject to} \\
\end{align*}
\[ o_i = \text{trace}(A_i X) + |c_i|^2 + (a_i x)^* c_i + (a_i x)^* c_i \]
where \( \parallel \cdot \parallel_2 \) returns the \( l_2 \) norm of vector and \( \delta \) is a design parameter. The optimization solution \( x \) can be achieved without phase ambiguity. Furthermore, if we know additional prior information about \( x \), for example, the bound of the real or imaginary part of each entry in \( x \), this prior information can be put into the optimization problem (27) as linear constraints,
\[
\begin{align*}
\text{minimize} \\
\text{subject to} \\
\end{align*}
\[ o_i = \text{trace}(A_i X) + |c_i|^2 + (a_i x)^* c_i + (a_i x)^* c_i \]
where \( \parallel \cdot \parallel_2 \) returns the \( l_2 \) norm of vector and \( \delta \) is a design parameter. The optimization solution \( x \) can be achieved without phase ambiguity. Furthermore, if we know additional prior information about \( x \), for example, the bound of the real or imaginary part of each entry in \( x \), this prior information can be put into the optimization problem (27) as linear constraints,
\[
\begin{align*}
\text{minimize} \\
\text{subject to} \\
\end{align*}
\[ o_i = \text{trace}(A_i X) + |c_i|^2 + (a_i x)^* c_i + (a_i x)^* c_i \]
where \( \parallel \cdot \parallel_2 \) returns the \( l_2 \) norm of vector and \( \delta \) is a design parameter. The optimization solution \( x \) can be achieved without phase ambiguity. Furthermore, if we know additional prior information about \( x \), for example, the bound of the real or imaginary part of each entry in \( x \), this prior information can be put into the optimization problem (27) as linear constraints,
\[
\begin{align*}
\text{minimize} \\
\text{subject to} \\
\end{align*}
\[ o_i = \text{trace}(A_i X) + |c_i|^2 + (a_i x)^* c_i + (a_i x)^* c_i \]
where \( \parallel \cdot \parallel_2 \) returns the \( l_2 \) norm of vector and \( \delta \) is a design parameter. The optimization solution \( x \) can be achieved without phase ambiguity. Furthermore, if we know additional prior information about \( x \), for example, the bound of the real or imaginary part of each entry in \( x \), this prior information can be put into the optimization problem (27) as linear constraints,
\[
\begin{align*}
\text{minimize} \\
\text{subject to} \\
\end{align*}
\[ o_i = \text{trace}(A_i X) + |c_i|^2 + (a_i x)^* c_i + (a_i x)^* c_i \]
where \( \parallel \cdot \parallel_2 \) returns the \( l_2 \) norm of vector and \( \delta \) is a design parameter. The optimization solution \( x \) can be achieved without phase ambiguity. Furthermore, if we know additional prior information about \( x \), for example, the bound of the real or imaginary part of each entry in \( x \), this prior information can be put into the optimization problem (27) as linear constraints,
polarization is measured. This hybrid target consists of a foam cylinder (SAITEC SBF 300) with diameter of 80 mm and relative permittivity of 1.45±0.15 as well as a plastic cylinder (berylon) with diameter of 31 mm and relative permittivity of 3±0.3 [19]. The frequency of the sounding signal is 2 GHz. The target domain is 150 mm by 150 mm. In order to control the number of variables to be solved in SDP, the imaging resolution is set to be 15 mm. The number of transmitter sensors is 8 and the number of receiver sensors is 241. The positions of transmitter sensors are taken with a step of \(3\) m from \(0\) to \(8\) and the number of receiver sensors is \(241\). The distance between transmitter sensor and the center of the target domain is \(1.67\) m. The distance between the center of the target domain and receiver sensor is also \(1.67\) m. The positions of transmitter sensors are taken with a step of \(45^\circ\) from \(0^\circ\) to \(60^\circ\) [19]. The positions of receiver sensors are taken with a step of \(1^\circ\) from \(0^\circ\) to \(60^\circ\) [19]. For \(\Delta x = \chi_{r_d} k^2 \Delta, n_d = 1, 2, \ldots, N_d\) where \(\chi_{r_d}\) is the contrast function; \(k\) is the wavenumber related to the sounding signal; \(\Delta\) is the area of the subarea in the target domain. The imaging result using accurate full-data scattered fields by Born iterative method with Tikhonov regularization is shown in Fig. 1. The imaging result using accurate amplitude-only total fields by the proposed novel single-step approach is shown in Fig. 2. The imaging results illustrate the acceptable performance of the proposed approach.

**VI. CONCLUSION**

This letter has presented a novel single-step approach for self-coherent tomography using semidefinite relaxation. No phase retrieval for scattered fields is required. Both man-made and experimental data are used to demonstrate the acceptable performance of the proposed approach. Mesh refinement to improve pattern reconstruction and algorithm speedup will be taken into account in the following paper. Meanwhile, we will continue to work on wireless tomography to push the integration of sensing and wireless communication systems.

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**REFERENCES**


