Solution of Nonlinear Equations: Graphical and Incremental Search Methods

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Solution of Nonlinear Equations

Introduction
- General Form of the Problem
- Types of Nonlinear Equations
- Graphical Interpretation

Example: Fluid Mechanics

Incremental Search Method

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Part I

Solution of Nonlinear Equations
Many engineering problems involve finding one or more values of $x$ that satisfy one of the following forms of equations:

1. **Form 1:**
   \[ f(x) = 0 \]

2. **Form 2:**
   \[ g(x) = C \\ f(x) = g(x) - C = 0 \]

3. **Form 3:**
   \[ g(x) = h(x) \\ f(x) = g(x) - h(x) = 0 \]
Types of Nonlinear Equations

- Polynomial equations
- Transcendental equations
  - Exponential equations
  - Logarithmic equations
  - Trigonometric equations
  - Hyperbolic equations
Solutions to equations of the form $f(x) = 0$ can be seen as places where the graph of $f(x)$ crosses or touches the $x$ axis.

![Graphical Interpretation of Nonlinear Equations](image)

**Figure 2.1**
$x_1, x_2, x_3, x_4$ - roots of the equation.
Graphical Interpretation

Solutions to equations of the form $f(x) = g(x)$ can be seen as places where the graphs of $f(x)$ and $g(x)$ intersect.
Water is discharged from a reservoir through a long pipe as shown. By neglecting the change in the level of the reservoir, the transient velocity of the water flowing from the pipe, $v(t)$, can be expressed as

$$
\frac{v(t)}{\sqrt{2gh}} = \tanh \left( \frac{t}{2L} \sqrt{2gh} \right),
$$

where $h$ is the height of the fluid in the reservoir, $L$ is the length of the pipe, $g$ is the acceleration due to gravity, and $t$ is the time elapsed from the beginning of the flow.
Governing Equations

\[ \frac{v(t)}{\sqrt{2gh}} = \tanh \left( \frac{t}{2L} \sqrt{2gh} \right) \]

Find the value of \( h \) necessary for achieving a velocity of \( v = 5 \) m/s at time \( t = 3 \) s when \( L = 5 \) m and \( g = 9.81 \) m/s\(^2\).
Solution of Equation

Substitute the values for $v$, $t$, $L$, and $g$ into the previous equation on the left side

$$\frac{v(t)}{\sqrt{2gh}} = \frac{5}{\sqrt{2(9.81)h}} = \frac{1.1288}{\sqrt{h}}$$

and the right side

$$\tanh \left( \frac{t}{2L} \sqrt{2gh} \right) = \tanh \left( \frac{3}{2(5)} \sqrt{2(9.81)h} \right) = \tanh \left( 1.3288\sqrt{h} \right)$$
Solution of Equation

Plot the two sides of the equation as separate functions of \( h \), then find their intersections. In this case, the two graphs intersect around \( h = 1.45 \) m, so the original equation is satisfied with \( h = 1.45 \) m.
Incremental Search Method

Incremental search is the most basic automated numerical method for solving nonlinear equations. The method:

1. Pick a starting point $x_0$ and a step size $\Delta x$. Use a positive $\Delta x$ if you want to search to the right, and a negative $\Delta x$ if you want to search to the left.

2. Let $x_1 = x_0 + \Delta x$ and calculate $f(x_0)$ and $f(x_1)$.

3. If the sign of $f(x)$ changes between $x_0$ and $x_1$, it is assumed that a root of $f(x)$ exists on the interval $(x_0, x_1)$.

4. If the sign of $f(x)$ does not change between $x_0$ and $x_1$, let $x_2 = x_1 + \Delta x$ and repeat the process.
Find the root of the equation

\[ f(x) = \frac{1.1288}{\sqrt{h}} - \tanh \left( 1.3288 \sqrt{h} \right) = 0 \]

using the incremental search method with \( x_0 = 1.0 \) and \( \Delta x = 0.1 \).

Evaluate the function \( f(x) \) at \( x = 1.0, 1.1, 1.2, \ldots \):

- \( x_0 = 1.0 \) \hspace{1cm} \( f(x_0) = 0.2598 \)
- \( x_1 = 1.1 \) \hspace{1cm} \( f(x_1) = 0.1923 \)
- \( x_2 = 1.2 \) \hspace{1cm} \( f(x_2) = 0.1336 \)
- \( x_3 = 1.3 \) \hspace{1cm} \( f(x_3) = 0.0822 \)
- \( x_4 = 1.4 \) \hspace{1cm} \( f(x_4) = 0.0366 \)
- \( x_5 = 1.5 \) \hspace{1cm} \( f(x_5) = -0.0040 \)
Since the sign of $f(x)$ changed between $x = 1.4$ and $x = 1.5$, we assume there is a root of $f(x)$ between 1.4 and 1.5. Repeating this method with $x_0 = 1.4$ and $\Delta x = 0.01$ would allow us to make a more accurate estimate of the root.
Incremental Search Limitations

- Only finds real-valued roots of $f(x)$. It cannot find complex roots of polynomials.
- Only finds roots where $f(x)$ crosses the $x$ axis. It cannot find roots where $f(x)$ is tangent to the $x$ axis.
- May be fooled by singularities in $f(x)$, such as in the tangent and cotangent functions.
- If the step size $\Delta x$ is too large, you may miss closely-spaced roots by skipping over them.
Example of Singularities

\[ f(x) = \cot x \]
Homework

- Read articles on course homepage labeled “Lec. 01 Reading”.
- Read and work along with examples given in Chapters 1–2 of *Getting Started with MATLAB 7* (link on course homepage).
- Be prepared to receive homework assignments on bisection and Newton-Raphson methods Monday.