Appendix A. Circular Tridiagonal Matrix Algorithm

Figure below shows N number of control volumes (2, 3, ..., N-1, N, N+1) plus two additional control volumes (i=1 and N+2).

![Diagram of control volumes](image)

Dependent variable $\phi$ are to be determined for N control volumes ($2 \leq i \leq N+1$) by solving simultaneous equations

$$A_i \phi_i = B_i \phi_{i-1} + C_i \phi_{i-1} + D_i$$  \hspace{1cm} (1)

Periodic boundary conditions are

$$\phi_1 = \phi_{N-1}$$

$$\phi_{N-2} = \phi_2$$  \hspace{1cm} (2)

Note that due to the periodic boundary condition, control volume thickness for i=1 and i=N+2 is not zero anymore. Control volume size for i=1 is equal to that of i=N+1 and i=i+2 equals i=2.

Consider Eq.(1) for i=2,

$$A_2 \phi_2 = B_2 \phi_3 + C_2 \phi_1 + D_2$$  \hspace{1cm} (3)

But $\phi_1 = \phi_{N-1}$ from periodic boundary condition. Thus replacing $\phi_1$ and rearranging, we have

$$\phi_2 = E_2 \phi_3 + F_2 \phi_{N-1} + G_2$$  \hspace{1cm} (4)

where

$$E_2 = \frac{B_2}{A_2}$$

$$F_2 = \frac{C_2}{A_2}$$

$$G_2 = \frac{D_2}{A_2}$$  \hspace{1cm} (5)
For $i=3$, Eq. (1) is

$$A_3 \phi_3 = B_3 \phi_4 + C_3 \phi_2 + D_3$$

Replacing $\phi_2$ in this equation with Eq. (4), we have

$$\phi_3 = E_3 \phi_4 + F_3 \phi_{N-1} + G_3$$  \hspace{1cm} (6)

where

$$E_3 = \frac{B_3}{A_3 - C_3 E_2} \quad ; \quad F_3 = \frac{C_3 F_2}{A_3 - C_3 E_2} \quad ; \quad G_3 = \frac{C_3 G_2 + D_3}{A_3 - C_3 E_2}$$  \hspace{1cm} (7)

Likewise,

$i = 4$

$$\phi_4 = E_4 \phi_5 + F_4 \phi_{N-1} + G_4$$  \hspace{1cm} (8)

where

$$E_4 = \frac{B_4}{A_4 - C_4 E_3} \quad ; \quad F_4 = \frac{C_4 F_3}{A_4 - C_4 E_3} \quad ; \quad G_4 = \frac{C_4 G_3 + D_4}{A_4 - C_4 E_3}$$  \hspace{1cm} (9)

$i = i$

$$\phi_i = E_i \phi_{i-1} + F_i \phi_{N-1} + G_i$$  \hspace{1cm} (10)

where

$$E_i = \frac{B_i}{A_i - C_i E_{i-1}} \quad ; \quad F_i = \frac{C_i F_{i-1}}{A_i - C_i E_{i-1}} \quad ; \quad G_i = \frac{C_i G_{i-1} + D_i}{A_i - C_i E_{i-1}}$$  \hspace{1cm} (11)

$i = N$

$$\phi_N = E_N \phi_{N-1} + F_N \phi_{N-1} + G_N$$  \hspace{1cm} (12)

where

$$E_N = \frac{B_N}{A_N - C_N E_{N-1}} \quad ; \quad F_N = \frac{C_N F_{N-1}}{A_N - C_N E_{N-1}} \quad ; \quad G_N = \frac{C_N G_{N-1} + D_N}{A_N - C_N E_{N-1}}$$  \hspace{1cm} (13)

and they are all known quantities.

If $\phi_{N-1}$ is known, then $\phi_N$ can be calculated from Eq. (12) and backward substitution yields $\phi_{N-1}$. 

$\hspace{1cm} A - Z$
... $\phi_4$, $\phi_3$, and $\phi_2$.

To find $\phi_{N-1}$, return to Eq. (1). At $i=N+1$, we have

$$A_{N-1} \phi_{N-1} = B_{N-1} \phi_{N-2} + C_{N-1} \phi_N + D_{N-1}$$

But $\phi_{N-2} = \phi_2$. Replacing $\phi_2$ with Eq. (4) and rearranging, we have

$$(A_{N-1} - B_{N-1} F_2) \phi_{N-1} = B_{N-1} E_2 \phi_3 + B_{N-1} G_2 + C_{N-1} \phi_N + D_{N-1}$$  \hspace{1cm} (14)

Let

$$p_2 = A_{N-1}$$
$$q_2 = B_{N-1}$$
$$r_2 = D_{N-1}$$  \hspace{1cm} (15)

Then Eq. (14) can be rewritten as

$$(p_2 - q_2 F_2) \phi_{N-1} = q_2 E_2 \phi_3 + q_2 G_2 + r_2 + C_{N-1} \phi_N$$  \hspace{1cm} (16)

Let

$$p_3 = p_2 - q_2 F_2 \hspace{0.5cm}; \hspace{0.5cm} q_3 = q_2 E_2 \hspace{0.5cm}; \hspace{0.5cm} r_3 = q_2 G_2 + r_2$$  \hspace{1cm} (17)

Then Eq. (16) becomes

$$p_3 \phi_{N-1} = q_3 \phi_3 + r_3 + C_{N-1} \phi_N$$  \hspace{1cm} (18)

Substituting Eq. (6) into Eq. (18), and rearranging, we have

$$p_4 \phi_{N-1} = q_4 \phi_4 + r_4 + C_{N-1} \phi_N$$  \hspace{1cm} (19)

where

$$p_4 = p_3 - q_3 F_3 \hspace{0.5cm}; \hspace{0.5cm} q_4 = q_3 E_3 \hspace{0.5cm}; \hspace{0.5cm} r_4 = q_3 G_3 + r_3$$  \hspace{1cm} (20)

Repeating this process, we have at $i = i$,

$$p_i \phi_{N-i} = q_i \phi_i + r_i + C_{N-1} \phi_N$$  \hspace{1cm} (21)

where

$$p_i = p_{i-1} - q_{i-1} F_{i-1} \hspace{0.5cm}; \hspace{0.5cm} q_i = q_{i-1} E_{i-1} \hspace{0.5cm}; \hspace{0.5cm} r_i = q_{i-1} G_{i-1} + r_{i-1}$$  \hspace{1cm} (22)

At $i = N$
\[ p_N \phi_{N-1} = q_N \phi_N + r_N + C_{N-1} \phi_N \]  \hspace{1cm} (23)

Substituting Eq. (12) for \( \phi_N \) and rearranging we get

\[ p_N \phi_{N-1} = (q_N + C_{N-1})(E_N + F_N)\phi_{N-1} + (q_N + C_{N-1})G_N + r_N \]  \hspace{1cm} (24)

Solving for \( \phi_{N-1} \), we finally obtain

\[ \phi_{N-1} = \frac{(q_N + C_{N-1})G_N + r_N}{p_N - (q_N + C_{N-1})(E_N + F_N)} \]  \hspace{1cm} (25)

Summary of CTDMA procedure:

(1) Define

\[ E_2 = \frac{B_2}{A_2} ; \quad F_2 = \frac{C_2}{A_2} ; \quad G_2 = \frac{D_2}{A_2} \]

(2) Evaluate for \( 3 \leq i \leq N \)

\[ E_i = \frac{B_i}{A_i - C_i E_{i-1}} ; \quad F_i = \frac{C_i F_{i-1}}{A_i - C_i E_{i-1}} ; \quad G_i = \frac{C_i G_{i-1} + D_i}{A_i - C_i E_{i-1}} \]

(3) Define

\[ p_2 = A_{N-1} ; \quad q_2 = B_{N-1} ; \quad r_2 = D_{N-1} \]

(4) Evaluate for \( 3 \leq i \leq N \)

\[ p_i = p_{i-1} - q_{i-1} F_{i-1} ; \quad q_i = q_{i-1} E_{i-1} ; \quad r_i = q_{i-1} G_{i-1} + r_{i-1} \]

(5) Calculate \( \phi_{N-1} \) using Eq. (25)

(6) Backward substitution for \( N \geq i \geq 2 \) using Eq. (10)