Bisection and Newton-Raphson Methods

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Review of Previous Lecture

Bisection and Newton-Raphson Methods

Part I

Review of Previous Lecture

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- Sample problems solved with numerical methods
  - Natural frequencies of a vibrating bar
  - Static analysis of a scaffolding
  - Critical loads for buckling a column
  - Realistic Design Properties of Materials
- Solution of nonlinear equations
  - Introduction
  - Example: fluid mechanics
  - Incremental search method

Part II

Bisection and Newton-Raphson Methods

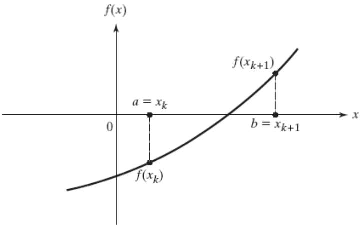
Bisection Method

One problem with the incremental search method is its lack of efficiency in finding a root. If a root is expected on the interval $0 < x < 1$, it will require between 1 and 10 loops through the method to bracket the root with 0.1 uncertainty:

- Calculate $f(0.0)$,
- Calculate $f(0.1)$,
- ...
- Calculate $f(1.0)$

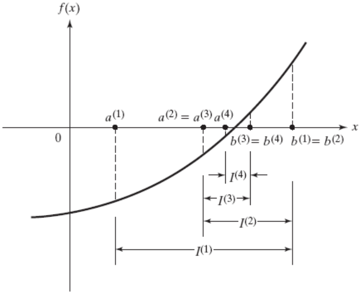
You may get lucky and only have to perform two function evaluations. You may be unlucky and have to perform 11 evaluations. In general, you'll probably have to perform 6-7 evaluations to find the solution. There has to be a more efficient way to find a solution.

Problem Setup



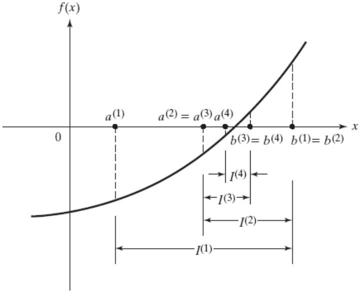
Start with a function $f(x)$ and two values of x (a and b) such that $f(a)$ and $f(b)$ have opposite signs. These values of a and b may be the final interval of an incremental search method with a relatively large step size.

Bisection Method Procedure



- Evaluate $f(x)$ at the midpoint of the interval, at $x_{\text{mid}} = \frac{a+b}{2}$.
- If $f(x_{\text{mid}}) \neq 0$, then the sign of $f(x_{\text{mid}})$ will match the sign of $f(a)$ or the sign of $f(b)$.
    - If $f(x_{\text{mid}})$ matches the sign of $f(a)$, then set $a = x_{\text{mid}}$ and repeat.
    - If $f(x_{\text{mid}})$ matches the sign of $f(b)$, then set $b = x_{\text{mid}}$ and repeat.

Bisection Method Procedure



Eventually, this method will limit the root of $f(x)$ to a sufficiently small interval, or $|f(x_{\mathrm{mid}})| \leq \epsilon$, where ϵ is the error tolerance for the problem.

Bisection Method Advantages

Since the bisection method discards 50% of the current interval at each step, it brackets the root much more quickly than the incremental search method does.

To compare:

- On average, assuming a root is somewhere on the interval between 0 and 1, it takes 6–7 function evaluations to estimate the root to within 0.1 accuracy.
- Those same 6–7 function evaluations using bisection estimates the root to within $\frac{1}{2^4} = 0.625$ to $\frac{1}{2^5} = 0.031$ accuracy.

Bisection Method Disadvantages

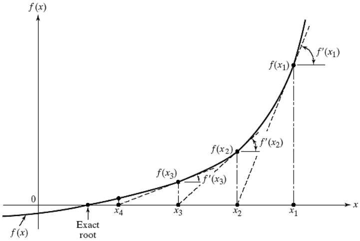
- Like incremental search, the bisection method only finds roots where the function crosses the x axis. It cannot find roots where the function is tangent to the x axis.
- Like incremental search, the bisection method can be fooled by singularities in the function.
- Like incremental search, the bisection method cannot find complex roots of polynomials.

Bisection Method Example

Find the root of $f(x) = x^3 - 2$ on the interval where $a = 1$ and $b = 2$, and $\epsilon = 0.05$:

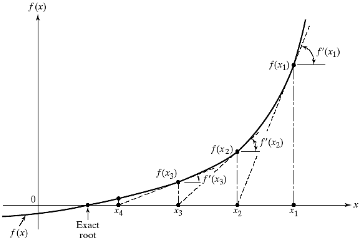
- $f(1) = 1^3 - 2 = -1$, $f(2) = 2^3 - 2 = 6$, $f(1.5) = 1.5^3 - 2 = 1.375$. Since 1.375 and 6 have the same sign, the root must be between 1 and 1.5.
- $f(1) = -1$, $f(1.5) = 1.375$, $f(1.25) = -0.047$. Since $|-0.047| < \epsilon$, the root is assumed to be at $x = 1.25$.
- If we used incremental search to find the same root, we would have required 4 function evaluations using a step size of 0.1, followed by 6 function evaluations using a step size of 0.01.

Problem Setup



Given: a function $f(x)$, its derivative $f'(x)$, a starting point x_1, and an error tolerance ϵ. We assume that both the height of the function $f(x)$ and its slope can help us make a more educated guess of the root x^*.

Newton-Raphson Method Procedure



Draw a line tangent to the function at the point $(x_1, f(x_1))$. The point where the tangent line crosses the x axis should be a better estimate of the root than x_1. Call that point x_2.

Calculate $f(x_2)$, and draw a line tangent to the function at the point $(x_2, f(x_2))$. The point where the new tangent line crosses the x axis should be a better estimate of the root than x_2. Call that point x_3.

Repeat until $|f(x)| < \epsilon$.

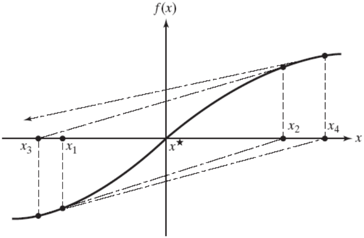
Newton-Raphson Method Advantages

- Unlike the incremental search and bisection methods, the Newton-Raphson method isn't fooled by singularities.
- Also, it can identify repeated roots, since it does not look for changes in the sign of $f(x)$ explicitly.
- It can find complex roots of polynomials, assuming you start out with a complex value for x_1.
- For many problems, Newton-Raphson converges quicker than either bisection or incremental search.

Newton-Raphson Method Disadvantages

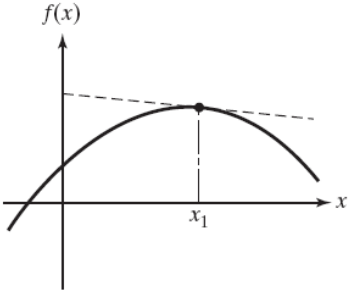
The Newton-Raphson method only works if you have a functional representation of $f'(x)$. Some functions may be difficult to impossible to differentiate. You may be able to work around this by approximating the derivative $f'(x) \approx \frac{f(x+\Delta x)-f(x)}{\Delta x}$.

Newton-Raphson Method Disadvantages



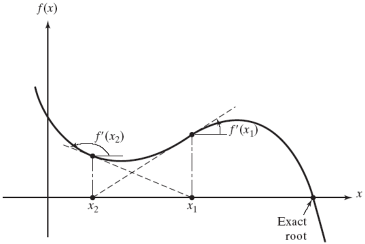
The Newton-Raphson method is not guaranteed to find a root. For example, if the starting point x_1 is sufficiently far away from the root for the function $f(x) = \tan^{-1} x$, the function's small slope tends to drive the x guesses further and further away from the root.

Newton-Raphson Method Disadvantages



If the derivative of the function at any tested point x_i is sufficiently close to zero, the next point x_{i+1} will be very far away. You may still find the root, but you will be delayed.

Newton-Raphson Method Disadvantages



If the derivative of the function changes sign near a tested point, the Newton-Raphson method may oscillate around a point nowhere near the nearest root.

Homework

Find the solution of $f(x) = x^2 - 10 = 0$ using the following methods and starting points:

- Bisection method, with $a = 1$, $b = 3$, and $\epsilon = 0.01$.
- Newton-Raphson method, with $x_0 = 0$ and $\epsilon = 0.01$.

Will both of these problem statements yield a solution? If not, what can you do to change the problem setup to allow you to find a solution?