Pulse Distortion Caused by Cylinder Diffraction and Its Impact on UWB Communications

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Abstract—One of the characteristics of UWB signals is pulse distortion, inherently determined by its huge bandwidth. Using cylinder model as an example, pulse distortion and its impacts on UWB system performance have been investigated. Although a lot of papers addressed the pulse distortion issue, quantifying the impacts of pulse distortion on the system performance appears to be novel. The simulation result shows that a SNR loss caused by template mismatch could reach as high as 4dB. It is also found that the range error caused by pulse distortion is much larger than the Cramer Rao Lower Bound (CRLB) and thus another source of errors fundamentally limiting the accuracies of time of arrivals (TOA) of a received signal. These results have direct applications in timing synchronization and positioning.

I. INTRODUCTION

Ultra-wideband (UWB) radio has attracted substantial attention recently due to its unique features [1]. Emerging applications of UWB are foreseen for sensor networks that are critical to mobile computing [2] [3]. Such networks combining low medium rate communications (50kbps to 1Mbps) over distances of 100 meters with positioning capabilities allow a new range of applications [2], including military applications, medical applications (monitoring of patients), family communications/supervision of children, search-and-rescue (communications with fire fighters, or avalanche/earthquake victims), control of home applications, logistics (package tracking), and security applications (localizing authorized persons in high-security areas).

When a sensor is placed in different environments, the non-line-of-sight (NLOS) propagation is encountered very often. Sometimes the propagation path is blocked by objects that can be modeled by a cylinder [4] [12]. For example, when a hill is smooth and not covered by trees or houses, the diffraction process is described more accurately in terms of creeping rays [4]. The purpose of this paper is to model such an environment, analyzing the possible pulse distortion and its impact on the system performance.

To be mathematically tractable, a simple channel consisting of a perfectly electrically conducting (PEC) cylinder is considered. The transceivers are placed such that only diffracted rays are present at the receiver. Noticing that the research of UWB sensors are still in its early stage, such a mathematically tractable physics-based channel model, although simple, may still give us a lot of insight.

In this paper, first, an impulse response for such a cylinder channel was derived in closed form. It is shown that the waveform of the received pulse is much different from that of the transmitted pulse. Since most of traditional receivers assume distortionless propagation, this pulse distortion will, naturally, cause degradation in terms of system performance. We then fully investigated the impacts of the pulse distortion on the system performance. Although a lot of papers have addressed the pulse distortion issue [8], and pulse distortion (frequency dependence) has been adopted by the IEEE 802.15.4a [6], quantifying the impact of pulse distortion on the system performance appears to be novel. In particular, we find that the SNR loss and timing error caused by pulse distortion in the correlation based receivers could be significant and thus deserve special attention.

The rest of the paper is organized as follows. Section II analyzes the impacts of the pulse distortion on the system performance. A closed form impulse response for a cylinder channel is derived in Section III, based on the well known frequency domain results. Some numerical results on pulse distortion and its impacts on system performance are shown in Section IV. Finally, conclusions are drawn in Section V.

II. PERFORMANCE DEGRADATION DUE TO PULSE DISTORTION

A. Impact of Pulse Distortion on System Performance

We follow the general system model and its performance expression in [5] [16]. When 0 and 1 are sent with equal probability, the average error probability in the receiver can be expressed as

\[ P_e = Q(SNR) \]  

where \( Q(x) \) is defined as

\[ Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} \exp(-y^2/2)dy \]

and \( SNR \) is the signal-to-noise ratio at the input to the threshold device, given by

\[ SNR = \frac{\|s_0(t) * q(t)\|_{t=T_0} - \|s_1(t) * q(t)\|_{t=T_0}}{\sqrt{2N_0}||q||} \]  

Here, \( s_0(t), s_1(t) \) are the received signals, which can be singular but their energies are limited, and \(*\) denotes convolution operation. Let \( p_0(t) \) and \( p_1(t) \) denote the transmitted signals,
and $h(t)$ denote the Channel Impulse Response (CIR), then it follows that $s_0(t) = p_0(t) * h(t)$ and $s_1(t) = p_1(t) * h(t)$. In (2), $q(t)$ is the local template used in the correlation based receiver, and $||q|| = [\int_{-\infty}^{\infty} q^2(t) dt ]^{1/2}$ is the norm of $q(t)$. It is known that for the optimum receiver, $q(t)$ is selected to be matched to the received signal $s_i(t)$. However, sometimes $q(t)$ is matched to the transmitted waveform: $p_0(t)$ and $p_1(t)$. This implies that the signal waveforms will not change as they pass through the channel. This mismatch practice will result in performance degradation in terms of $SNR$, which will be illustrated by numerical results.

B. Impact of Pulse Distortion on Timing Synchronization and Positioning

For a single-path AWGN channel, it can be shown that the best achievable accuracy of a position estimate $\hat{d}$ derived from TOA estimation satisfies the following inequality [2]

$$\sqrt{\text{var}(\hat{d})} \geq \frac{c}{2\sqrt{2\pi}SNR\beta}$$

(3)

where $c$ is the speed of light, $SNR$ is the signal-to-noise ratio, and $\beta$ is the effective (or root mean square) signal bandwidth defined by

$$\beta \triangleq \left[ \int_{-\infty}^{\infty} f^2 |P(f)|^2 df / \int_{-\infty}^{\infty} |P(f)|^2 df \right]^{1/2}$$

(4)

where $P(f)$ is the Fourier transform of the transmitted signal. For example, with a received bandwidth of 1.5 GHz, an accuracy of less than one inch can be obtained at $SNR = 0$ dB. However, sometimes the accuracy of TOA estimate is dominated by practical limitations such as clock synchronization (clock jitter). A new limitation caused by pulse distortion is first discovered in this paper. Note that the mismatch between the local template and the receiving waveform will cause a timing error, as well as a amplitude error of the correlation peak. In some conditions, as illustrated in the later numerical results, pulse distortion may become another factor affecting the accuracy of TOA estimate.

III. UWB CHANNEL MODELING FOR CYLINDER DIFFRACTION

Consider a simple environment that only consists of a cylinder, as illustrated in Fig. 1. The transmitter is located far from the cylinder and the receiver is within the shadow of the cylinder. We will firstly start from the frequency domain, borrowing some well-known results from the radio propagation literature. Then the inverse Laplace transform is applied to obtain the time domain impulse response. It should be noted that not all the frequency domain results can be transformed to the time domain. It is thus fortunate that the impulse response of the cylinder environment could be derived analytically.

A. Frequency Domain Channel Model

As shown in Fig. 1, the incident plane wave with unit amplitude is normally incident upon a Perfectly Electric Conductor (PEC) circular cylinder. The incident wave has a field component $V_z^i$ in the z-direction. Then we have $V_z^i = \exp(-jk\rho \cos\phi)$, where $k$ is wave number, and $P(\rho, \phi)$ is the coordinate of the receiver. Assuming electric polarization ($V_z^2 = E_z^2$), and $ka >> 1$, where $a$ is the radius of the cylinder, the electric field at the receiver within the shadow area is given as [13]

$$E_z(j\omega) = \sum_{n=1}^{\infty} \frac{D_n(j\omega)\exp\{-jk\rho \Omega_n(j\omega)l_1\}}{\exp\{-jk\rho \Omega_n(j\omega)l_2\}}$$

(5)

where $r = AP = B'P$ is the tangent distance between the receiver and the cylinder; $l_1$ and $l_2$ are the arc lengths, $l_1 = AA', l_2 = BB'$; $A, B$ are the glancing points of the incident plane wave on the cylinder. Let $\theta_1$ and $\theta_2$ denote the corresponding angles, respectively, then we have $l_1 = \theta_1 a$, $l_2 = \theta_2 a$. The attenuation constant $\Omega_n(j\omega)$ in (5) is given by

$$\Omega_n(j\omega) = \frac{\alpha_n}{a} \frac{\exp\left\{-\frac{\pi}{6}\right\}}{\sin\left\{\frac{\pi}{6}\right\}}$$

(6)

while the amplitude weighting factor $D_n(j\omega)$ in (5) is given by

$$D_n(j\omega) = 2\Omega_n'(-\alpha_n)^2 \left(\frac{ka}{2}\right) \exp\left\{-\frac{\pi}{6}\right\}$$

(7)

where $-\alpha_n$ are the zeros of the Airy function $Ai(.)$. In (7), $Ai'(.)$ represents the first derivative of the Airy function. The zeros of the Airy function and associated values used in our simulation can be found in [13] or regular mathematical handbooks.

In (5) the arc lengths $l_1$ and $l_2$ can be calculated by the following equation:

$$l_{1,2} = \left(\frac{\pi}{2} + \phi - \cos^{-1}\frac{\Omega_n}{\rho}\right)a$$

(8)

Assuming that $l_{1,2}$ is non-zero, which requires that the location of the receiver be in the shadow, we can use the first $N$ terms to represent the electric field $E_z$. Practically, considering that the Airy function roots, $-\alpha_n$, increase with $n$, the first several terms will be enough for most of the cases when the receiver is in the deep shadow.

Fig. 1. Plane wave diffraction by a circular cylinder.
It should be noted that an exact solution of the received field at $P(\rho, \phi)$ should also include the contributions from infinite number of circling rays. For large $ka$, however, these contributions are negligibly small and thus could be rightly ignored.

After some mathematical manipulation, Eq. (5) can be further expressed in a more convenient form as

$$E_z(j\omega) = \sum_{n=1}^{N} \left[ \exp\{-(jk + \Omega_n(j\omega))l_1 - jkr\} + \exp\{-(jk + \Omega_n(j\omega))l_2 - jkr\} \right] \frac{D_n(j\omega)}{\sqrt{8\pi jkr}}$$  \hspace{1cm} (9)

where $s = j\omega$, $A_n = \frac{(a/2c)^{3/2} \{A'i(-\alpha_n)^2\}}{\sqrt{2\pi\rho/c}}$, and $c$ is speed of light. The propagation delay terms, derived in (A-2) and (A-3), respectively, can be expressed as

$$\exp\{-(jk + \Omega_n(j\omega))l_1 - jkr\} = B_1(s) \exp(-\beta^1_n s^{3/2}) \hspace{1cm} (12)$$

$$\exp\{-(jk + \Omega_n(j\omega))l_2 - jkr\} = B_2(s) \exp(-\beta^2_n s^{3/2})$$  \hspace{1cm} (11)

where

$$\beta^1_n = \frac{\alpha_n}{a/2c} \cdot l_1, \hspace{1cm} \beta^2_n = \frac{\alpha_n}{a/2c} \cdot l_2, \hspace{1cm} (13)$$

Note that notation “1,2” in $\beta^1_n$ is superscript, not the power. Eq. (9) can be rewritten as

$$E_z(s) = \sum_{n=1}^{N} A_n s^{3/2} \left[ B_1(s) \exp(-\beta^1_n s^{3/2}) + B_2(s) \exp(-\beta^2_n s^{3/2}) \right]$$  \hspace{1cm} (14)

For the early time approximation, it follows that

$$L^{-1} \left\{ s^{3/2} \exp(-\alpha s) \right\} \sim \frac{\alpha^{3/2} \exp[-2t^{-\frac{1}{4}}(\alpha/3)^{3/2}]}{2\sqrt{\pi}t} u(t)$$  \hspace{1cm} (15)

where $L^{-1} \{ \}$ represents the inverse Laplace transform and $u(t)$ is the Heaviside’s step function. The transform pair in (15) has been first derived by Friedlander [15] and later proved by [14] through a numerical approach.

Taking inverse Laplace transform, the time domain (TD) field is obtained. Since the incident field is $V_z = \exp(-jk\rho\cos\phi)$, which corresponds to a delta function in the time domain, the time domain version of the received field $E_z$ in (9) can be viewed as the impulse response of the channel. Thus, we have

$$h(t) = \sum_{n=1}^{N} A_n [h^1_n(t) * \delta(t - t_1) + h^2_n(t) * \delta(t - t_2)]$$  \hspace{1cm} (16)

where

$$A_n = 2^{-\frac{3}{4}} \left( \frac{\pi \sqrt{\rho^2 - a^2}}{c} \right)^{-\frac{3}{4}} \left( \frac{a}{c} \right)^{3/2} \{A'i(-\alpha_n)^2\}^{-2}$$  \hspace{1cm} (17)

$$h^1_n(t) = \frac{1}{2} \sqrt{\beta_1^2} \exp[-2t^{-\frac{1}{4}}(\beta^1_n/3)^{3/2}] u(t)$$  \hspace{1cm} (18)

$$h^2_n(t) = \frac{1}{2} \sqrt{\beta_2^2} \exp[-2t^{-\frac{1}{4}}(\beta^2_n/3)^{3/2}] u(t)$$  \hspace{1cm} (19)

$$t_1 = \frac{l_1}{c} + \frac{\sqrt{\rho^2 - a^2}}{c} \hspace{1cm} t_2 = \frac{l_2}{c} + \frac{\sqrt{\rho^2 - a^2}}{c}$$  \hspace{1cm} (20)

It is worth noticing that the subscript $n$ in (16) is the number of terms used to approximate the impulse response, not the path number. In (16), there are two paths (rays), distinguished by the different timing delays $t_1$ and $t_2$.

The impulse response of cylinder can also be reformed as the generalized multipath model [5] [7] [9] as

$$h(t) = \sum_{l=1}^{2} h_l(t) * \delta(t - t_l)$$  \hspace{1cm} (21)

where $h_1(t) = \sum_{n=1}^{N} A_n h^1_n(t)$, and $h_2(t) = \sum_{n=1}^{N} A_n h^2_n(t)$.

we define the distortion parameter $h^m(t)$ for the m-th path as

$$h^m(t) = \frac{1}{2} \sqrt{\beta^m_3} \exp[-2t^{-\frac{1}{4}}(\beta^m_n/3)^{3/2}] u(t)$$  \hspace{1cm} (22)

Note that $h^m(t)$ can also be viewed as the diffraction factor of the creeping wave.

IV. NUMERICAL RESULTS AND ANALYSIS

A. Comparison of Frequency and Time Domain Results

Widely accepted frequency domain results will serve as the reference for the validity of our derived time domain expressions. In our simulation, the transmitted pulse $p(t)$ can be arbitrary, but a special pulse that is well suited for pulse distortion analysis is used here. This pulse is defined in the frequency domain and thus it is convenient to control the spectrum.

Frequency domain of the incident pulse $P(\omega)$ is defined by [14]

$$P(\omega) = C_0(1 - e^{-\omega T}) P_1 e^{-\omega P_2 T}$$  \hspace{1cm} (23)

where $T = \frac{1}{2\pi f_c} \ln \left( \frac{P_3 + P_5}{P_1 + P_3} \right)$, $f_c$ represents the center frequency, which can be adjusted easily according to different requirements. The peak of $P(\omega)$ is normalized by choosing $C_0 = \left( \frac{P_3 + P_5}{P_1 + P_3} \right) P_1$ and $P_2$. Then $p(t)$ is obtained by taking Inverse Fast Fourier transform (IFFT). The received signal can be calculated by using equation $s(t) = p(t) * h(t)$. Parameters
used for the transmitted pulse in our simulation are as follows: \( P_1 = 2, P_2 = 1, \) and \( f_c = 2 \text{GHz}. \)

Before studying received signal \( s(t) \), let us analyze the impulse response \( h(t) \) first. As we can see from (16), \( h_n^m(t) \) is the key term which causes pulse distortion. Our first step is to investigate the distortion effect of \( h_n^m(t) \). Since \( \beta_n^m \) is a constant for the specific \( n \) and \( m \), for simplicity, we let \( \beta_n^m = 1 \).

Fig. 2 shows the pulse shape distorted by \( h_n^m(t) \) with \( \beta_n^m = 1 \). The solid curve labeled by “TD” denotes the received waveform, which is obtained by convolving \( p(t) \) with \( h_n^m(t) \) in time domain directly; The “FD+IFFT” curve is obtained by taking IFFT of product \( P(\omega)H_n^m(\omega) \), where \( H_n^m(\omega) = (j\omega)^{-5}\exp[-\beta_n^m(j\omega)^2]\). The original transmitted pulse \( p(t) \) has been plotted and normalized to the “TD” result to compare the waveforms. As shown in Fig. 2, the pulse shape is severely distorted. This distortion appears in two aspects: one is that the pulse shape is different; the other one is that pulse is widened by the channel. Fig. 2 also shows that the time domain result agrees well with the frequency domain result, which validates our derivation in Section III.

After obtaining some insight from a simple case, a more complicated one is analyzed. Our final interest is in calculating the strength and shape of a pulse after being diffracted by the cylinder and analyzing the effect of distortion on the system design. Using the same pulse \( p(t) \), we analyze the received signal at point \( P(\rho, \phi) \). Parameters used in our simulation are as follows: \( \rho = 2.5m, \) a = 1.5m, and \( \phi = \pi/20 \). After the above parameter values are used into (8) (20), the time delays \( t_1 \) and \( t_2 \) can be calculated. Knowing \( \alpha_n \) and \( A^i(-\alpha_n) \), coefficients \( A_n, h_n^1, \) and \( h_n^2 \) are calculated by (17), (18), (19), respectively. Let \( N = 10 \), which means using the first ten terms to approximate the impulse response \( h(t) \).

Fig. 3 shows the received signal \( s(t) \). The solid curve labeled by “TD” is obtained by convolving \( p(t) \) with \( h(t) \) described in (16) with \( N = 10 \) in time domain. The “FD+IFFT” curve is obtained by taking IFFT of frequency domain product \( P(\omega)H(j\omega) \), where \( H(j\omega) = E_z(s) \) is calculated by Eq. (9).

It is observed that the “FD+IFFT” curve agrees with the “TD” curve very well, implying that our time domain derivation of the impulse response \( h(t) \) in Section III is correct.

It is also shown in Fig. 3 that the received signal strength is greatly reduced, due to the diffraction by a cylinder. The strength of the diffraction signal is about 8% of the signal incident on the cylinder. Besides the amplitude information, which is widely used in the traditional frequency domain framework to predict the path loss due to the diffraction, time domain framework also provides wave shape information. It should be noticed that for a UWB pulse, the shape of the received pulse is oftenly different from that of the transmitted pulse, as we can see from Fig. 3. These results are unusual compared with narrowband signals. In the following, we will analyze the effect of this distortion on the system performance.

### B. The Effect of Pulse Distortion on System Performance and Timing Errors

All the above sections show that a pulse is distorted after diffraction by a PEC cylinder. Due to the presence of pulse distortion, the interest of this section is to analyze how pulse distortion will affect the system performance, if there is no corresponding compensation.

Notice that the received signal \( s(t) \) in Fig. 3 consists of two paths, the stronger path signal will be chosen to study pulse distortion in the following analysis. Let \( s(t) \) denote the received pulse corresponding to the stronger path, then \( s(t) \) can be easily calculated by equation \( s(t) = p(t) * h_1(t) \). Different local templates of \( q(t) \) are then selected to be correlated with the signal \( s(t) \), as done in a correlation based receiver.

Fig. 4 shows the correlation results with different local templates. In Fig. 4, the dashed curve is the result of correlating \( s(t) \) with \( q(t) = p(t) \). Here, the transmitted pulse \( p(t) \)
has been used as the local template, which is the case in a narrowband communication system. The solid curve represents the result of correlating $s(t)$ with itself $q(t) = s(t)$, where the received signal $s(t)$ is used as the local template. Different templates have been normalized to have the same energy. This normalization facilitates the comparison of results. As shown in Fig. 4, the peak in the dashed curve is lower than that of the solid curve, and the corresponding timings are different. As explained in Section II, this distortion effect causes performance degradation in terms of SNR and timing. The peak of the symmetric waveform correctly gives the position while the peak of the asymmetric waveform gives an error in the position. The difference between these two peaks defines the timing error (also shown in Table I).

Fig. 5 shows the BER curves with different templates. SNR is calculated by (2). The dotted curve is obtained with $q(t) = p_0(t) - p_1(t)$, which uses the transmitted signals to form the local template. The solid curve is obtained with $q(t) = s_0(t) - s_1(t)$, using the received signals to form the local template. It is observed that at $BER = 10^{-3}$, a $4dB$ loss in SNR is caused by the unmatched local templates.

Due to the facts that the transfer function, say, Eq. (14), is frequency selective, pulse distortion is closely related to the signal bandwidth. It follows that the system performance depends on the bandwidths of the different signals. In the simulations carried for this paper, the pulse bandwidth can be adjusted conveniently by these pulse parameters $P_1, P_2$ and $f_c$ in (23). Table I shows the SNR loss and timing errors caused by pulse distortion. Different transmitted pulses with different bandwidths have been tested. RMS bandwidth is calculated by using (4). From Table I, it is observed that the mismatch is proportional to signal bandwidth in a nonlinear manner, implying that bandwidth is not the only cause affecting pulse distortion.

The impact of template mismatch is significant to timing acquisitions and positioning, as pointed out in Section II(B). Notice that one nanosecond in the timing error roughly transforms into one foot in the positioning error. The CRLB is calculated using (3) with SNR=0 dB. In Table I it is observed that the ranging error caused by template mismatch is much bigger than the CRLB—the best achievable accuracy of position estimate. The position error caused by template mismatch is one bottleneck for accurate positioning. Notice that NLOS environment are dealt with in this paper. NLOS paths result in an error in positioning that usually requires a LOS path. This is a very difficult problem for a long time. Comparing this NLOS error with the error caused by a mismatched template is beyond the scope of this paper.

V. CONCLUSION

The extreme bandwidth of a UWB pulse signal causes pulse distortion due to the frequency selectivity of attenuation in the channel. This paper advances the prior art by developing an analytical framework to model the scenario in which a UWB pulse is diffracted by a PEC circular cylinder, where the radius of the cylinder is at least several times larger than the wavelengths of the frequencies contained in a UWB pulse.

When a pulsed plane wave is normally incident on a cylinder and the observer (receiver) lies in the shadow region, closed form expressions for the impulse response of this channel are derived for the first time. The analytical work has its intrinsic value from a view of propagation theory, although limited applications have been demonstrated in this paper.

The applications of these analytical results are significant in several aspects. First the possible impacts of the cylinder channel on system performance is fully investigated. A 4 dB loss in SNR has been observed at $BER=10^{-3}$ if the waveform of the local template in a general correlation based receiver is mismatched to the waveform of the received signal. Second, pulse distortion caused by the cylinder channel introduces timing errors in a correlation based receiver. In particular, it is discovered that the error caused by pulse distortion in a mismatched template (mentioned above) is much larger than...
the CRLB and thus one of the bottlenecks in achieving the accuracy of TOA estimate in positioning. The results in this paper thus suggest that it is problematic to use the waveform of the transmitted signals to form the local template in a general correlation based receiver. This result justifies the efforts to understand the physical mechanisms in different pulse propagation environments and their impacts on system designs. In the future more channels that are mathematically tractable need to be investigated.

ACKNOWLEDGMENT

The first author would like to thank Martha A. Calderon for her careful revision of the language of this manuscript.

APPENDIX

TRANSFORM OF EQ. (9) INTO A FORM SUITABLE FOR LAPLACE TRANSFORM

Let \( s = j \omega \), the diffraction term in Eq. (9) can be rewritten as

\[
\frac{D_n(j\omega)}{\sqrt{8\pi \kappa r}} = \frac{2A\exp\left(-\frac{\alpha}{2\kappa} s^\frac{1}{2}\right)}{\sqrt{8\pi \kappa r}} \exp\left[s \frac{1}{2} - \frac{1}{2}\left(\frac{a}{c}\right)^2 \frac{1}{s}ight] = \frac{\sqrt{2\pi r/c}}{2} \left\{A\left(-\alpha s\right)^{-1/2}\right\} \frac{1}{s^{1/2}}
\]

where \( A_n = \frac{(a/2c)^2}{\sqrt{2\pi r/c}} \left\{A\left(-\alpha s\right)^{-1/2}\right\} \frac{1}{s^{1/2}} \).

The propagation term in Eq. (9) can be rewritten as

\[
\exp\left\{-\left(jk + \Omega_n(j\omega)l1 - jkr\right)\right\} = \exp\left\{-\left.jk(l1 + r) - \frac{1}{\alpha} \left(\frac{a}{2c}\right)^2 \frac{1}{s}\right\} \exp\left(j\pi/2\right) \frac{1}{s} l1\right\} = \exp\left\{-\left.jk(l1 + r)\right\} \exp\left(-\frac{1}{\alpha} \left(\frac{a}{2c}\right)^2 \frac{1}{s} l1\right\} = B_1(s) \exp\left(-\beta_n^2 s^2\right)
\]

where \( B_1(s) = \exp\left(-s(l1 + r)/c\right) \), and \( \beta_n = -\frac{1}{\alpha} \left(\frac{a}{2c}\right)^2 l1 \).

Similarly, it follows that

\[
\exp\left\{-\left.jk + \Omega_n(j\omega)l2 - jkr\right)\right\} = \exp\left\{-s(l2 + r)/c\right\} \exp\left(-\frac{1}{\alpha} \left(\frac{a}{2c}\right)^2 l2 s^\frac{1}{2}\right) = B_2(s) \exp\left(-\beta_n^2 s^2\right)
\]

where \( B_2(s) = \exp\left(-s(l2 + r)/c\right) \), and \( \beta_n^2 = -\frac{1}{\alpha} \left(\frac{a}{2c}\right)^2 l2 \).

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