Part V: Velocity and Acceleration Analysis of Mechanisms

This section will review the most common and currently practiced methods for completing the kinematics analysis of mechanisms; describing motion through velocity and acceleration. This section of notes will be divided among the following topics:

1) Overview of velocity and acceleration analysis of mechanisms
2) Velocity analysis: analytical techniques
3) Velocity analysis: Classical techniques (instant centers, centrodes, etc.)
4) Static force analysis, mechanical advantage
5) Acceleration analysis: analytical techniques
6) Acceleration analysis, Classical techniques

1) Overview of velocity and acceleration analysis of mechanisms:

Important features associated with velocity and acceleration analysis:

a. Kinematics: 
\[ \text{motion} \rightarrow X, \quad v = \frac{dx}{dt}, \quad a = \frac{dv}{dt} \]

b. Types of equations that result
\[ \text{Linear} \]

c. General approach/strategy
\[ \text{modeling} \rightarrow \text{eq}s. \quad \text{solve position w/} \quad \text{w.r.t. time} \quad \text{velocity eq}s. \quad \text{solve as linear eq}s. \]

d. Uses/Applications
- Use velocity or acceleration.
- Dynamics.
2) Velocity Analysis: Analytical Techniques

The standard approach to velocity analysis of a mechanism is to take derivative of the position equations w.r.t. time. (Note, alternative approaches, such as those termed influence coefficients, can be performed by first taking the partial derivative with respect to an alternate parameter multiplied by the time derivative of that parameter)

The position equations that we consider are predominantly loop closure and constraint equations. The approach will take the derivatives of these equations, expand into scalar equations and solve for the unknowns (all problems will be linear in the unknowns!).

Example:

1) Derivative of a loop equation:

\[ \dot{\mathbf{X}} = \dot{\mathbf{V}} = \frac{d\mathbf{X}}{dt} - \frac{d\mathbf{X}}{d\theta} \cdot \frac{d\theta}{dt} \]

2) Move knowns to one side of equation, unknowns to the other side

3) Expand into scalar equations:

4) Cast into matrix form and solve:
Review: Taking time derivatives of a vector.

In cartesian vector notation:
\[ \ddot{r} = \ddot{r} \hat{r} + \dot{r} \mathbf{v} = \frac{\partial}{\partial t}(\ddot{r}) = \frac{\partial}{\partial t}(\dot{r}) \]

Expand into scalar components:
Explanation of Velocity Terms – or – The Dynamics of the Dukes:

For a loop equation:

\[
\vec{r}_2 + \vec{r}_3 = \vec{r}_1 + \vec{r}_4
\]

\[
\dot{\vec{r}}_2 e^{i\theta} + r_2 i \dot{\theta} e^{i\theta} + \dot{\vec{r}}_3 e^{i\theta} + r_3 i \dot{\theta} e^{i\theta} = \dot{\vec{r}}_1 e^{i\theta} + r_1 i \dot{\theta} e^{i\theta} + \dot{\vec{r}}_4 e^{i\theta} + r_4 i \dot{\theta} e^{i\theta}
\]

\[
\frac{d(F)}{dt} = \frac{d}{dt}(\vec{r}_1 e^{i\theta} + r_1 i \dot{\theta} e^{i\theta})
\]

\[
V_{dukes} = \frac{d}{dt}(\vec{r}_1 e^{i\theta})
\]
Performing Velocity Analysis – The Process:

Step 1: Get velocity eq’s → derivatives of position eq’s.

Step 2: Write velocity eq’s as scalar eq’s.

Step 3: Cast eq’s in matrix form.

\[ A \ddot{v} = \dddot{b} \]

Step 4: Solve for \( u \)’s as

\[ \ddot{v} = A^{-1} \dddot{b} \]

\[ \uparrow \text{Software} \]
Example 1: Velocity analysis

1. Review vector model and position analysis:
2. From mobility: \( n=6, f_1=7, f_2=0 \Rightarrow M=1 \) This means there is one input, so input velocity and acceleration rates are known (for the hydraulic cylinder)
3. Create vector model (See fig.)
4. Count unk’s: First, velocity: \( q_2a, q_2b, q_3, q_4, r_5, q_5 \) = 6 variables, \( r_5 \) = input.
5. Take derivatives of position equations, Solve
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Step 2: Matrix Form

\[ \begin{bmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \\ \mathbf{e} & \mathbf{f} & \mathbf{g} & \mathbf{h} \\ \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{l} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} \mathbf{m} \\ \mathbf{n} \\ \mathbf{o} \end{bmatrix} \]

Step 3: Coefficients, \( \mathbf{a}_x = \mathbf{e}_z \), \( \mathbf{a}_y = \mathbf{f}_z \), \( \mathbf{a}_z = \mathbf{g}_z \)

Step 4: \( \mathbf{x} = \mathbf{b}_z \), \( \mathbf{y} = \mathbf{c}_z \), \( \mathbf{z} = \mathbf{d}_z \)

Step 5: \( \mathbf{u} \equiv \mathbf{v} \equiv \mathbf{w} \)

\[ \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{bmatrix} = \begin{bmatrix} \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \end{bmatrix} \]

Step 6: \( \mathbf{u} = \mathbf{A}_z \), \( \mathbf{v} = \mathbf{B}_z \), \( \mathbf{w} = \mathbf{C}_z \)

Step 7: Final Answer

\[ \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_z \\ \mathbf{B}_z \\ \mathbf{C}_z \end{bmatrix} \]
Example 3:
3) Velocity analysis: Classical techniques (instant centers, centrodes, etc.)

Classical techniques for velocity analysis consisted predominantly of graphical techniques and determination of instant centers. The graphical techniques involved drawing velocity polygons that form geometric equivalents of our derivative loop closure equations. Due to the fact that analytical techniques can be easily programmed and formalized, graphical techniques have been largely outdated. However, some of these techniques provide a significant amount of insight into the problem and will be reviewed briefly here. The techniques reviewed are:

- Instant Centers
- Centrodes

Instant Centers of Velocity or instant centers are a point common to two bodies which has the same velocity for both bodies (at that instant in time). The number of instant centers for an n-body linkage is:

\[ c = n(n-1)/2. \]

The instant center of two links connected by a revolute is trivial (it is that revolute). The instant center for two links connected by a slider is also simple (the center of curvature of the slider axis).

Kennedy’s Theorem: Any three bodies in plane motion will have three instant centers and they will lie on the same straight line.
Applying Kennedy’s theorem to a four bar:

Applying Kennedy’s theorem to a Slider-crank:

A few examples of the use of instant centers:
**Centrodes:**
A Centrode is the curve defined by the locations of the instant center over the range of motion of a mechanism. Each possible instant center can create two centrodes, found by considering the motion relative to each of the two links defining the instant center. One centrode will be called the fixed centrode and one a moving centrode. The centrodes can then recreate the motion of the fourbar by rolling (without slip) in contact with each other. As an example, fourbars can be used to define the profile for non-circular gears (for example elliptical gears).
4) **Mechanical Advantage (and first steps toward Kinetic Force Analysis):**

With the ability to analyze the velocity of a mechanism, a kinetic force analysis can be directly performed. This process is based on the principles of conservation of energy (or power here since we assume the constraints are not time-dependent) and superposition.

First, consider Mechanical Advantage, the ratio of output force to input force. For a conservative system, (not a bad assumption for a well-designed mechanism), the power into the mechanism equals the power out:

\[
P_{\text{in}} = P_{\text{out}}
\]

With the power instantaneously defined as,

\[
P = \frac{F \cdot V}{T} = \frac{F}{T} \cdot V
\]

Then, the Mechanical Advantage is defined as:

\[
\eta = \frac{\vec{F}_o \cdot \vec{V}_o}{\vec{F}_i \cdot \vec{V}_i}
\]

Assumes

\[
\vec{F}_o \cdot \vec{V}_o \text{ in some direction}
\]

Assumes

\[
\vec{F}_i \cdot \vec{V}_i \text{ in some direction}
\]

![Mechanical Advantage Diagram]

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Example: Kinetic Force Analysis
Kinetic Force Analysis:
To complete the kinetic force analysis, first apply the idea of mechanical advantage to evaluate the input force required for each applied load. If loads are given on multiple links, then evaluate the mechanical advantage for these multiple links. Second, apply the principle of superposition (these problems are linear in force). Thus, the total input force is given as the superposition of the input forces required for each of the applied loads.

The last possible step to consider here is constraint forces (forces in the bearings). In line with the concept of static force analysis based on conservation of energy, one approach would be to repeat the process above for every bearing, but instantaneously eliminate the motion constraint from each bearing, and solve for the force required to enforce this constraint. For example, to find the x-component reaction of a bearing, allow that bearing to move (assign it unit velocity) instantaneously (i.e., bearing does not change position). Solve for the mechanical advantage relating the x-force at that bearing to all applied loads, and sum to get the total x-directed reaction via superposition. This method is known as the method of Lagrange Multipliers. If only one or two reactions are desired, it is relatively easy to apply. If all reactions are desired, it makes more sense to apply the techniques of kinetostatic analysis (to be covered in upcoming topics).
5) Acceleration Analysis: Analytical Techniques:

As in velocity analysis, the acceleration analysis of a mechanism is performed by taking the derivative of the position equations w.r.t. time. When acceleration analysis is performed, it is of particular interest to note the frame in which derivatives are taken, since Newton’s second equation requires acceleration with respect to an inertial frame (also called Newtonian frame). In our analytical acceleration analysis of mechanisms, we will ensure inertial accelerations by having our mechanism grounded to an appropriate inertial frame.

The second derivative of the position vector equations will yield vector equations in acceleration (second derivatives of constraint equations will yield scalar constraints in acceleration). The unknown accelerations are linear and are solved using linear algebra. The approach proceeds as follows:

1) Take second derivative of a loop equation:

2) Isolate unknowns on one side of equation knowns on the other

3) Expand into scalar equations:

4) Cast into matrix form and solve:

The first step is to review the second derivative of a vector.

\[
\dddot{\mathbf{r}} = \mathbf{r} \dddot{\mathbf{r}} = \mathbf{r} \dddot{\mathbf{e}}^i
\]

\[
\dddot{a} = \frac{d^2}{dt^2} (\dddot{\mathbf{r}}) = \frac{d^2}{dt^2} (\mathbf{r} \dddot{\mathbf{e}}^i)
\]
\[
\dddot{\mathbf{a}} = \frac{d}{dt} (\dddot{\mathbf{r}}) + \dddot{\mathbf{r}} \hat{\omega} \times \dddot{\mathbf{r}} + \dddot{\hat{\omega}} \times \mathbf{r} + \hat{\omega} \times \dddot{\mathbf{r}} + \dddot{\mathbf{r}} \hat{\omega} \times \dddot{\mathbf{r}}
\]
\[
\dddot{\mathbf{a}} = \frac{d}{dt} (\dddot{\mathbf{r}}) \mathbf{e}^i + \dddot{\mathbf{r}} \frac{d}{dt} (\hat{\mathbf{e}}) \mathbf{e}^i + \hat{\mathbf{e}} \frac{d}{dt} (\mathbf{r} \hat{\mathbf{e}}) \mathbf{e}^i + \hat{\mathbf{e}} \frac{d}{dt} (\mathbf{r} \hat{\mathbf{e}}) \mathbf{e}^i + \hat{\mathbf{e}} \frac{d}{dt} (\mathbf{r} \hat{\mathbf{e}}) \mathbf{e}^i
\]

\[
\dddot{\mathbf{a}} = \dddot{\mathbf{r}} \mathbf{e}^i + \dddot{\hat{\mathbf{e}}} \mathbf{r} \mathbf{e}^i - \dddot{\mathbf{r}} \mathbf{e}^i + 2 \dddot{\mathbf{r}} \mathbf{e}^i
\]

\[
\dddot{\mathbf{a}} = \dddot{\mathbf{r}} \mathbf{e}^i + \dddot{\hat{\mathbf{e}}} \mathbf{r} \mathbf{e}^i - \dddot{\mathbf{r}} \mathbf{e}^i + 2 \dddot{\mathbf{r}} \mathbf{e}^i
\]
Expand into scalar components:

Cartesian:
\[ \ddot{a} = \ddot{r} \hat{r} + \dddot{r} \times \hat{r} + \ddot{\omega} \times \dot{r} + 2 \dot{\omega} \times \ddot{r} \]

Complex Polar:
\[ \ddot{a} = \dot{r} e^{i\theta} + \ddot{r} i e^{i\theta} - \dot{\theta}^2 r e^{i\theta} + 2 \dot{\theta} \dot{r} e^{i\theta} \]

Explanation of Acceleration Terms – or – The Dynamics of the Dukes:
For a loop equation:

\[ r_2 + r_3 = r_1 + r_4 \]

More discussion of acceleration: Examples of each term in the acceleration equation.

- Linear acceleration term
- Angular acceleration term
- Centripetal acceleration term
- Coriolis acceleration term
Performing Acceleration Analysis – The Process:
Example 1: Acceleration on 3-pt hitch:

Example 1 (cont.)
Example 2:
6) Acceleration analysis: Classical techniques (acceleration polygon method)

The classical techniques for acceleration analysis consists predominantly of graphical techniques, creating a vector acceleration polygon to describe acceleration of multiple points (and their relative accelerations). From any given polygon, two unknown scalar components in acceleration can be determined (much like our loop equations), and so the problem proceeds. With the ability to solve systems of equations on computers easily and accurately, this method is no longer used other than to provide intuition on the behavior of a system.